**Cynthia Furse**
University of Utah
Dept. of Electrical & Computer
Engineering, Rm. 3280 MEB
50 S. Central Campus Dr.
Salt Lake City, UT 84112-9206 USA
Tel: +1 (801) 581-6941
Fax: +1 (801) 581-5281
E-mail: cfurse@ece.utah.edu

# On the Equivalence of Geometrical-Optics Reflected Fields and the Stationary-Phase Solution of the Associated Radiation Integrals

*A. J. Booysen*

[redacted]

## Abstract

This tutorial shows how the Geometrical-Optics expression for the field reflected from a surface can be derived directly from the general first-order stationary-phase solution of the associated radiation integral. It is a purely mathematical exercise, but hopefully will improve the understanding of how these two techniques are related to one another. The material presented here is ideally suited for post-graduate courses on high-frequency electromagnetics techniques.

Keywords: Geometrical optics; electromagnetic scattering; electromagnetic radiation; electromagnetic reflection; stationary phase; radiation integral; physical optics

## 1. Introduction

The high-frequency electromagnetic field reflected from a surface can be expressed in terms of Geometrical-Optics type rays, similar to optical rays reflected from a surface. The Geometrical-Optics field expression is usually derived from conservation of energy of a wave propagating away from a source [1]. The scattered field can also be calculated by means of the radiation integrals, for which the equivalent current on the reflector has to be calculated in some way or another. This is typically done by numerical techniques, such as the Method of Moments [2]. When the reflector is large and smooth in terms of wavelength, one can employ asymptotic techniques to obtain analytical solutions for the radiation integrals, such as the familiar first-order stationary-phase solution of the integral [3].

Although it is common knowledge that the Geometrical-Optics reflected field and the first-order stationary-phase solution of the associated radiation integral must yield the same result, it is instructive to see how these expressions can, in general, be mathematically related to each other. This tutorial will show, step-by-step, how the stationary-phase expression can be manipulated to yield the corresponding Geometrical-Optics field expression. It will become evident that Geometrical Optics goes hand-in-hand with another high-frequency approximation, the so-called Physical Optics approach [1].

# 2. Stationary Phase Solution in Vector Notation

Consider the two-dimensional scattering problem of Figure 1, in which an electric line source illuminates a perfectly conducting surface, $S$, described by a vector, $\mathbf{r}$. The observation point is denoted by the vector $\mathbf{r}_2$, the source point by $\mathbf{r}_1$, and the distance between the point on $S$, described by $\mathbf{r}$ and the observation point, is $R = |\mathbf{r}_2 - \mathbf{r}|$.

For a $TM_z$ incident field,

$$\mathbf{E}^i = \frac{e^{-jkl_1}}{\sqrt{l_1}} \hat{\mathbf{z}} = E_z^i \hat{\mathbf{z}}. \tag{1}$$

The Geometrical-Optics reflected field is given by [1]

$$\mathbf{E}^i = -\frac{e^{-jkl_1}}{\sqrt{l_1}} \sqrt{\frac{\rho}{\rho + l_2}} e^{-jkl_2} \hat{\mathbf{z}}, \tag{2}$$

where

$$\frac{1}{\rho} = \frac{1}{l_1} + \frac{2}{r_c \cos\theta^i}, \tag{3}$$

and $r_c$ is the radius of curvature, $\rho$ is the caustic distance, $\theta^i$ is the angle of incidence at the reflection point, and the other relevant parameters are as indicated in Figure 1.

The scattered electric field can also be calculated by means of the radiation integrals. The electromagnetic field illuminating the reflector induces a surface current, $\mathbf{J}$, on the equivalent reflector surface. The scattered electric field for the incident field specified by Equation (1) can then be calculated from

$$\mathbf{E}^s = E_z \hat{\mathbf{z}} = \frac{-k\eta}{4} \int_S J_z \hat{\mathbf{z}} H_0^{(2)}(kR)\, ds, \tag{4}$$

where $\mathbf{J} = J_z \hat{\mathbf{z}}$, $R = l_2$ is the distance from the integration point on the equivalent reflector surface to the field-evaluation point, $H_0^{(2)}$ is a Hankel function of the second type, and $ds$ is the incremental integration arc length.

For large distances away from $S$, the large-argument form of the Hankel function can be used [4], and Equation (4) becomes

$$\mathbf{E}^s = E_z \hat{\mathbf{z}} = -\eta \sqrt{\frac{k}{8\pi}} e^{j\frac{\pi}{4}} \hat{\mathbf{z}} \int_S J_z \frac{e^{-jkR}}{\sqrt{R}} ds. \tag{5}$$

If the radius of curvature of $S$ is large in terms of the wavelength, the electric surface current induced on $S$ is, according to the Physical-Optics approach [1],

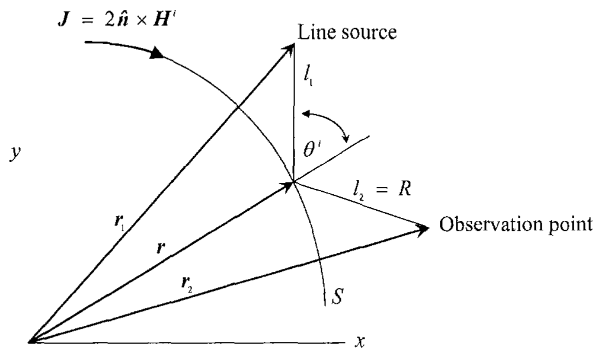



**Figure 1. The geometry and vectors used for the two-dimensional analysis of reflection from the perfectly conducting surface, $S$, illuminated by a line source. J is the equivalent electric current.**

$$\begin{aligned}
\mathbf{J} &\approx 2\hat{\mathbf{n}} \times \mathbf{H}^i \\
&= 2\hat{\mathbf{n}} \times \left( \hat{\mathbf{l}}^i \times \frac{1}{\eta} \mathbf{E}^i \right) \\
&= \frac{2E_z^i}{\eta} \left[ \hat{\mathbf{n}} \times \left( \hat{\mathbf{l}}^i \times \hat{\mathbf{z}} \right) \right] \\
&= \frac{2E_z^i}{\eta} \left( \hat{\mathbf{n}} \cdot \hat{\mathbf{l}}^i \right) \hat{\mathbf{z}} \\
&= J_z \hat{\mathbf{z}}.
\end{aligned} \tag{6}$$

Equation (5) can then be written as

$$\mathbf{E}^s = E_z \hat{\mathbf{z}} = \sqrt{\frac{k}{2\pi}} e^{j\frac{\pi}{4}} \hat{\mathbf{z}} \int_S \frac{\hat{\mathbf{n}}' \cdot \hat{\mathbf{l}}^{i\prime}}{\sqrt{l_1' l_2'}} e^{-jk(l_1' + l_2')} ds'. \tag{7}$$

In Equation (7), the primed vectors indicate that they are a function of the integration coordinates $x$ and $y$. It should be noted that it is imperative that the Physical-Optics approach be used in order to obtain equivalence with the Geometrical-Optics field expression.

The vector $\mathbf{r}$ is expressed in the Cartesian coordinate system as $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and the integral in Equation (7) has to be evaluated as such. If we let $S$ be represented in parametric format as $\mathbf{r} = \mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$, the integration in Equation (7) can be performed with respect to a single parameter, $t$, and Equation (7) can be written in the form [5]

$$\int_S \frac{\hat{\mathbf{n}}' \cdot \hat{\mathbf{l}}^{i\prime}}{\sqrt{l_1' l_2'}} e^{-jk(l_1' + l_2')} ds' = \int_S F(t) e^{jk\Psi(t)} dt, \tag{8}$$

where $ds'/dt = \sqrt{\dot{\mathbf{r}} \bullet \dot{\mathbf{r}}} = |\dot{r}|$, and $\dot{\mathbf{r}}$ is the derivative of $\mathbf{r}$ with respect to $t$. In Equation (8), we have

$$F(t) = \frac{\hat{\mathbf{n}}' \bullet \hat{\mathbf{i}}^{i'}}{\sqrt{l_1' l_2'}} \left| \frac{d\mathbf{r}}{dt} \right| = \frac{\hat{\mathbf{n}}' \bullet \hat{\mathbf{i}}^{i'}}{\sqrt{l_1' l_2'}} |\dot{\mathbf{r}}|, \tag{9}$$

and

$$\Psi(t) = -(l_1' + l_2'). \tag{10}$$

The primed vectors, $\hat{\mathbf{n}}'$, $\hat{\mathbf{i}}^{i'}$, and the quantities $l_1'$, $l_2'$, and $ds'$ in the preceding equations are all functions of the parameter $t$, and can be expressed as

$$\hat{\mathbf{n}}' = \frac{\dot{\mathbf{r}}' \times (\mathbf{r}' \times \dot{\mathbf{r}}')}{|\dot{\mathbf{r}}' \times (\mathbf{r}' \times \dot{\mathbf{r}}')|}, \tag{11}$$

$$\mathbf{l}_1' = \mathbf{r}' - \mathbf{r}_1', \tag{12}$$

$$\mathbf{l}_2' = \mathbf{r}_2' - \mathbf{r}', \tag{13}$$

$$l_1' = |\mathbf{l}_1'|, \tag{14}$$

$$l_2' = |\mathbf{l}_2'|, \tag{15}$$

and

$$\hat{\mathbf{i}}^{i'} = \hat{\mathbf{l}}_1' = \frac{\mathbf{l}_1'}{|\mathbf{l}_1'|}. \tag{16}$$

The integral in Equation (8) has a stationary point, $t_s$, defined by

$$\frac{d\Psi(t_s)}{dt} = \dot{\Psi}(t_s) = 0, \tag{17}$$

which corresponds to the reflection point on $S$. To denote evaluation at the stationary point, all the primed quantities in Equations (9) to (16) are changed to unprimed quantities. The stationary-phase solution of the right-hand integral in Equation (8) is [3]

$$I \sim F(t_s) \sqrt{\frac{2\pi}{k\ddot{\Psi}(t_s)}} e^{jk\Psi(t_s)} e^{j\frac{\pi}{4}}, \tag{18}$$

where $\ddot{\Psi}(t_s)$ is the second derivative of $\Psi$ with respect to $t$, evaluated at the stationary point, and it was assumed that $\ddot{\Psi}(t_s) > 0$. Equation (5) can thus be expressed as

$$\begin{aligned} \mathbf{E}^s = E_z \hat{\mathbf{z}} &\sim \sqrt{\frac{k}{2\pi}} e^{j\frac{\pi}{4}} \sqrt{\frac{2\pi}{k\ddot{\Psi}(t_s)}} (\hat{\mathbf{n}} \bullet \hat{\mathbf{l}}_1) |\dot{\mathbf{r}}| \hat{\mathbf{z}} e^{jk\Psi(t_s)} e^{j\frac{\pi}{4}} \\ &\approx \frac{\hat{\mathbf{n}} \bullet \hat{\mathbf{i}}^i}{\sqrt{\ddot{\Psi}(t_s)}} \frac{|\dot{\mathbf{r}}|}{\sqrt{l_1 l_2}} \hat{\mathbf{z}} e^{-jk(l_1+l_2)} e^{j\frac{\pi}{2}}, \end{aligned} \tag{19}$$

which we want to prove to be identical to Equation (2). In order to do so, let

$$\begin{aligned} -\Psi(t_s) &= l_1 + l_2 \\ &= \sqrt{\mathbf{l}_1 \bullet \mathbf{l}_1} + \sqrt{\mathbf{l}_2 \bullet \mathbf{l}_2} \\ &= \sqrt{(\mathbf{r} - \mathbf{r}_1) \bullet (\mathbf{r} - \mathbf{r}_1)} + \sqrt{(\mathbf{r}_2 - \mathbf{r}) \bullet (\mathbf{r}_2 - \mathbf{r})}. \end{aligned} \tag{20}$$

The first derivative of Equation (20) with respect to $t$ at the stationary point is

$$-\dot{\Psi}(t_s) = \frac{\dot{\mathbf{r}} \bullet (\mathbf{r} - \mathbf{r}_1)}{\sqrt{(\mathbf{r} - \mathbf{r}_1) \bullet (\mathbf{r} - \mathbf{r}_1)}} - \frac{\dot{\mathbf{r}} \bullet (\mathbf{r}_2 - \mathbf{r})}{\sqrt{(\mathbf{r}_2 - \mathbf{r}) \bullet (\mathbf{r}_2 - \mathbf{r})}}, \tag{21}$$

and the second derivative can, after some manipulation, be shown to be

$$\begin{aligned} -\ddot{\Psi}(t_s) = &\frac{\ddot{\mathbf{r}} \bullet (\mathbf{r} - \mathbf{r}_1) + \dot{\mathbf{r}} \bullet \dot{\mathbf{r}} - \dfrac{[\dot{\mathbf{r}} \bullet (\mathbf{r} - \mathbf{r}_1)]^2}{(\mathbf{r} - \mathbf{r}_1) \bullet (\mathbf{r} - \mathbf{r}_1)}}{\sqrt{(\mathbf{r} - \mathbf{r}_1) \bullet (\mathbf{r} - \mathbf{r}_1)}} \\ &+ \frac{-\ddot{\mathbf{r}} \bullet (\mathbf{r}_2 - \mathbf{r}) + \dot{\mathbf{r}} \bullet \dot{\mathbf{r}} - \dfrac{-\dot{\mathbf{r}} \bullet (\mathbf{r}_2 - \mathbf{r})}{(\mathbf{r}_2 - \mathbf{r}) \bullet (\mathbf{r}_2 - \mathbf{r})}}{\sqrt{(\mathbf{r}_2 - \mathbf{r}) \bullet (\mathbf{r}_2 - \mathbf{r})}}. \end{aligned} \tag{22}$$

This can be rewritten in terms of the vector quantities of Equations (11) to (16) as

$$\begin{aligned} -\ddot{\Psi}(t_s) = &\frac{\ddot{\mathbf{r}} \bullet \mathbf{l}_1 + |\dot{\mathbf{r}}|^2 - \dfrac{[\dot{\mathbf{r}} \bullet \mathbf{l}_1]^2}{l_1^2}}{l_1} \\ &+ \frac{-\ddot{\mathbf{r}} \bullet \mathbf{l}_2 + |-\dot{\mathbf{r}}|^2 - \dfrac{[-\dot{\mathbf{r}} \bullet \mathbf{l}_2]^2}{l_2^2}}{l_2}. \end{aligned} \tag{23}$$

The vector $\dot{\mathbf{r}}$ is a vector tangential to $S$, as indicated in Figure 2, and therefore

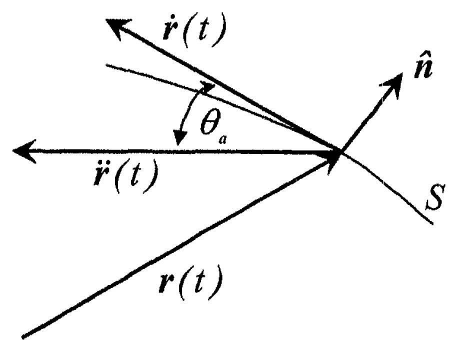



**Figure 2. The definition of the parametric vectors describing the contour $S$.**

$$\dot{\mathbf{r}}\bullet\mathbf{l}_1 = |\dot{\mathbf{r}}| l_1 \cos\left(\frac{\pi}{2} \pm \theta^i\right)$$
$$= \mp l_1 |\dot{\mathbf{r}}| \sin\theta^i, \quad (24)$$

so that

$$\frac{[\dot{\mathbf{r}}\bullet\mathbf{l}_1]^2}{l_1^2} = |\dot{\mathbf{r}}|^2 \sin^2\theta^i$$
$$= |\dot{\mathbf{r}}|^2 - |\dot{\mathbf{r}}|^2 \cos^2\theta^i. \quad (25)$$

When Equation (25) and an identical expression for the corresponding $l_2$ term are substituted into Equation (23), the result is

$$-\ddot{\Psi}(t_s) = \frac{\ddot{\mathbf{r}}\bullet\mathbf{l}_1 + |\dot{\mathbf{r}}|^2\cos^2\theta^i}{l_1} + \frac{-\ddot{\mathbf{r}}\bullet\mathbf{l}_2 + |\dot{\mathbf{r}}|^2\cos^2\theta^i}{l_2}, \quad (26)$$

which can be written as

$$-\ddot{\Psi}(t_s) = \ddot{\mathbf{r}}\bullet(\hat{\mathbf{l}}_1 - \hat{\mathbf{l}}_2) + |\dot{\mathbf{r}}|^2\cos^2\theta^i\left(\frac{1}{l_1} + \frac{1}{l_2}\right). \quad (27)$$

In Equation (27), the vectors $\hat{\mathbf{l}}_1$ and $\hat{\mathbf{l}}_2$ are unit vectors in the direction of $l_1$ and $l_2$, respectively. With Equation (27) and $\hat{\mathbf{n}}\bullet\hat{\mathbf{l}}_1 = \cos\theta^i$, it follows that

$$\frac{\hat{\mathbf{n}}\bullet\hat{\mathbf{l}}_1}{\sqrt{l_2\ddot{\Psi}(t_s)}} = \frac{-\cos\theta^i|\dot{\mathbf{r}}|}{\sqrt{l_2\left[\ddot{\mathbf{r}}\bullet(\hat{\mathbf{l}}_1 - \hat{\mathbf{l}}_2) + |\dot{\mathbf{r}}|^2\cos^2\theta^i\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right]}} e^{-j\frac{\pi}{2}}, \quad (28)$$

where the phase term on the right is due to the negative sign preceding $\ddot{\Psi}(t_s)$ in Equation (27). Furthermore, $\hat{\mathbf{l}}_1 - \hat{\mathbf{l}}_2 = -2\cos\theta^i\hat{\mathbf{n}}$ and $\ddot{\mathbf{r}}\bullet\hat{\mathbf{n}} = -|\ddot{\mathbf{r}}|\sin\theta_a$ (Figure 2), which, upon substitution in Equation (28) and subsequent simplification, yields

$$\frac{\hat{\mathbf{n}}\bullet\hat{\mathbf{l}}_1}{\sqrt{l_2\ddot{\Psi}(t_s)}} = -\left(\frac{2l_2\cos\theta^i|\ddot{\mathbf{r}}|\sin\theta_a}{|\dot{\mathbf{r}}|^2\cos^2\theta^i} + \frac{l_2}{l_1} + 1\right)^{-\frac{1}{2}} e^{-j\frac{\pi}{2}}. \quad (29)$$

From differential geometry [6], we have

$$r_c = \frac{|\dot{\mathbf{r}}|^3}{|\ddot{\mathbf{r}}\times\dot{\mathbf{r}}|} = \frac{|\dot{\mathbf{r}}|^2}{|\ddot{\mathbf{r}}|\sin\theta_a} \quad (30)$$

where $r_c$ is the radius of curvature of $S$ at the stationary point. With Equation (30) now substituted into Equation (29) and the result into Equation (19), further simplification yields

$$E^s = -\hat{\mathbf{z}}\frac{e^{-jkl_1}}{\sqrt{l_1}}\sqrt{\frac{\rho}{\rho + l_2}} e^{-jkl_2}, \quad (31)$$

which concludes the proof.

## 3. Conclusion

Although the above mathematical exposition does not present any new information, it is hoped that valuable insight will be gained into the relationship between the high-frequency Geometrical-Optics fields and their associated radiation integrals.

## 4. References


1. W. L. Stutzman and G. A. Thiele, *Antenna Theory and Design*, New York, John Wiley & Sons, 1981, p. 446-458.

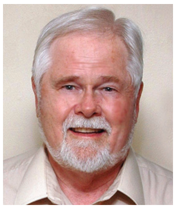
2. R. F. Harrington, *Field Computation by Moment Methods*, Kriegler Publishing Company, 1982.

3. L. B. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, Englewood Cliffs, NJ, Prentice-Hall, 1973, p. 370-416.

4. R. F. Harrington, *Time-Harmonic Electromagnetic Fields*, New York, McGraw-Hill, 1961, p. 463.

5. E. Kreyszig, *Advanced Engineering Mathematics*, New York, John Wiley & Sons, 1962, Fourth Edition, p. 411.

6. I. D. Faux and M. J. Pratt, *Computational Geometry for Design and Manufacture*, Ellis Horwood, 1979.



Raymond P. Wasky

## SHORT COURSES

### EW AGAINST THE NEW THREAT ENVIRONMENT

9–12 April 2019, Columbia, Maryland, United States. Applied Technology Institute, 349 Berkshire Drive, Riva, MD 21140-1433, United States, +1 888 501 2100 or +1 410 956 8805, fax: +1 410 956 5785, email: ati@ATIcourses.com. http://www.ATIcourses.com.

### BASIC ANTENNA CONCEPTS

9–11 April 2019, San Diego, California, United States.

### ANTENNA ENGINEERING

15–19 April 2019, Atlanta, Georgia, United States.

### BASIC RF ELECTRONIC WARFARE CONCEPTS

16–18 April 2019, Atlanta, Georgia, United States.

### PHASED-ARRAY RADAR SYSTEMS

16–18 April 2019, Atlanta, Georgia, United States. Georgia Institute of Technology, Professional Education, P.O. Box 93686, Atlanta, GA 30377-0686, United States, +1 404 385 3500, fax: +1 404 894 8925. http://www.pe.gatech.edu.

### MODELING AND SIMULATION OF ANTENNAS

7–10 May 2019, Atlanta, Georgia, United States.

### FUNDAMENTALS OF RADAR SIGNAL PROCESSING

13–16 May 2019, Denver, Colorado, United States.

### NEAR-FIELD ANTENNA MEASUREMENT TECHNIQUES

13–17 May 2019, Boulder, Colorado, United States. Georgia Institute of Technology, Professional Education, P.O. Box 93686, Atlanta, GA 30377-0686, United States, +1 404 385 3500, fax: +1 404 894 8925. http://www.pe.gatech.edu.

*Digital Object Identifier 10.1109/MAP.2019.2895619*
*Date of publication: 3 April 2019*

## CORRECTION

In the October 2002 issue of *IEEE Antennas and Propagation Magazine*, typographical errors slipped through when converting the original WordPerfect document to a Microsoft Word document [1], which was the only format acceptable for publication at that time.

*Digital Object Identifier 10.1109/MAP.2019.2895644*
*Date of publication: 3 April 2019*

Equation (2) should read as

$$\mathbf{E}^r = -\frac{e^{-jkl_1}}{\sqrt{l_1}}\sqrt{\frac{\rho}{\rho + l_2}}\, e^{-jkl_2}\hat{\mathbf{z}} \qquad (2)$$

and (31) should read as

$$\mathbf{E}^s = -\hat{\mathbf{z}}\frac{e^{-jkl_1}}{\sqrt{l_1}}\sqrt{\frac{\rho}{\rho + l_2}}\, e^{-jkl_2}. \qquad (31)$$

### REFERENCE

[1] A. J. Booysen, “On the equivalence of geometrical-optics reflected fields and the stationary-phase solution of the associated radiation integrals,” *IEEE Antennas Propag. Mag.*, vol. 44, no. 5, pp. 120–123, Oct. 2002.