Circular Aperture Pattern Synthesis From Collapsed Equivalent Line-Source Distributions

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***Abstract*—A novel circular aperture pattern synthesis technique is presented, which enables a linear line-source distribution to be converted to a rotationally symmetric circular aperture distribution, of which any ϕ-cut radiation pattern is ideally the same as the principal plane pattern of the line-source distribution. Line-source pattern synthesis techniques are numerous and versatile and the technique presented here allows these techniques to be applied to circular apertures as well. This new synthesis method is most compatible with line-source distributions which have zero edge illumination.**

***Index Terms*—Antenna radiation patterns, aperture antennas.**

I. Introduction

THIS PAPER presents a technique which allows symmetric line-source distributions to be converted to equivalent rotationally symmetric circular aperture distributions. The term 'equivalent' is used to indicate that the circular aperture distribution will ideally yield a rotationally symmetric radiation pattern identical to the principal plane radiation pattern of the line-source distribution. There are numerous techniques available for line-source pattern synthesis (e.g., [1]–[3]) and circular aperture pattern synthesis (e.g., [4]–[13]). Most of the earlier circular aperture synthesis techniques were limited to Taylor-type distributions for patterns with low sidelobes. The generalized circular aperture synthesis method later presented by Elliott and Stern [10] allows patterns of arbitrary shape and sidelobe levels to be synthesized. The synthesis method presented here offers an alternative method for the synthesis of circular aperture antenna patterns. The value of the new synthesis method lies not only in the simple conversion of existing line-source distributions to equivalent rotationally symmetric radial distributions, but it also reveals some important characteristics of circular aperture distributions. A novel application of Elliott and Stern's synthesis method for circular apertures is also introduced. It will be shown that their method can be used to synthesize linear distributions with the additional constraint of zero edge illumination, for patterns of arbitrary shape and sidelobe envelope.

The technique described in this paper is based on the collapsing of the circular aperture distribution to an equivalent line-source distribution. For circular aperture pattern synthesis the process is reversed and a numerical procedure is adopted to calculate the radial distribution which will yield a rotationally symmetric pattern which will ideally be identical to the principal plane pattern of the associated line-source distribution.

Manuscript received July 16, 2003; revised December 24, 2003.

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The method of collapsing an arbitrary aperture distribution to an equivalent line-source distribution is a known technique (see, for example, [14]) and will be discussed only briefly. It should become evident that any planar aperture with an arbitrary aperture distribution and boundary has a collapsed equivalent line-source (CELS) distribution in any plane perpendicular to the aperture. In the synthesis method adopted here for circular apertures, a known line-source distribution is regarded as the CELS distribution of the circular aperture. The rotationally symmetric radial distribution is then approximated by either a Fourier or a Taylor series expansion, or a combination of both. The unknown Fourier/Taylor coefficients are then solved for by means of matrix inversion. Several examples will be presented to verify the accuracy of the synthesis method.

II. Collapsing of Aperture Distributions

To visualize the process of collapsing the aperture distribution to an equivalent line-source distribution, it is perhaps best to consider the way in which the radiation pattern of a planar antenna array is calculated. For the rectangular array of $M' \times N'$ elements ($M' = 2M+1, N' = 2N+1$) with arbitrary excitation I_{mn} shown in Fig. 1, the radiated field strength at angle ϕ, θ is given by (see, for example, [15])

$$P(\theta,\phi) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} I_{mn} e^{jk \sin\theta (m d_x \cos\phi + n d_y \sin\phi)}. \quad (1)$$

For the sake of an example, let us consider the plane defined by $\phi = 0^\circ$ (any other value of ϕ will also suffice, provided that the excitation values are collapsed perpendicularly to the $z\phi$-plane). Recognising that the exponential term is the same for every column (all terms having the same m), (1) can be rearranged as

$$P(\theta,0) = \sum_{m=-M}^{M} a_m e^{jk \sin\theta (m d_x)} \quad (2)$$

where

$$a_m = \sum_{n=-N}^{N} I_{mn}. \quad (3)$$

Equation (2) presents the radiation pattern of a linear array of $M' = 2M+1$ elements, with the excitation of each element given by the sum of all $N' = 2N+1$ element excitations in that column ($M = 3$ and $N = 2$ in Fig. 1). It should be evident that it is much quicker to calculate the radiation pattern of the CELS distribution than the radiation pattern of the complete array, as the time required to calculate the exponential terms in (1) and (2)

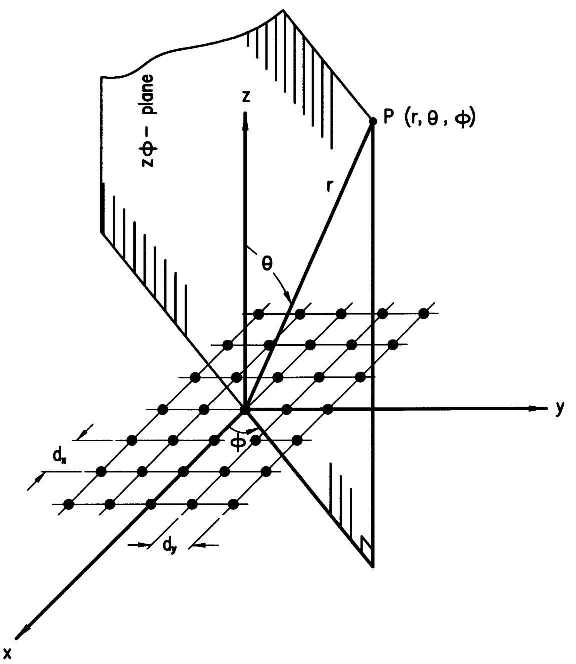


Fig. 1. Coordinate system for planar array comprised of isotropic radiators.

is significantly longer than the time required to add up a series of complex numbers. It must however be emphasized that the plane of cut must be perpendicular to the aperture of the antenna. The distribution is then collapsed straight down onto this plane. If the collapsing cannot be done along columns where the exponential terms are constant, not much advantage is to be gained in terms of time saving. In the design phase of a planar antenna array one is however at first usually only interested in the principal plane patterns and a significant time saving can be achieved by performing the collapsing of the columns. Other examples of where there should also be a time saving by collapsing of the aperture include the pattern calculation of cylindrical scanners (reducing the cylindrical surface integral to a single cylindrical contour integral) and during the design phase of symmetric reflector antennas, where the pattern calculation time can approximately be halved (every point in the lower half of the reflector has a symmetric point in the upper half for an azimuth plane cut).

For a circular aperture and in fact any aperture with a continuous distribution, the discrete summation process of an array becomes a continuous line integral. The CELS distribution $g(x)$ of a circular aperture can be calculated by means of the integral

$$g(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} I(r)\, dy \tag{4}$$

where $r = \sqrt{x^2+y^2}$ and $-R \leq x \leq R$. Equation (4) is the integral form of (3). When $I(r)$ is rotationally symmetric, (4) becomes

$$g(x) = 2\int_{0}^{\sqrt{R^2-x^2}} I(r)\, dy. \tag{5}$$

It should be evident that for $I(r)$ finite, $g(R)$ is always zero. This is an important observation because it puts a limitation on the line-source distributions which can be transformed to equivalent circular aperture distributions, ideally with negligible error. Many line-source distributions do not have zero edge illumination, whilst all circular aperture CELS distributions do.

For a constant aperture illumination $I(r) = 1$, (5) above reduces to

$$g(x) = 2\sqrt{R^2 - x^2}. \tag{6}$$

The radiation pattern for this CELS distribution (the factor 2 having been omitted) is given by

$$P(\theta) = \int_{-R}^{R} \sqrt{R^2 - x^2} e^{j\beta x \sin\theta}\, dx \tag{7}$$

where $\beta = 2\pi/\lambda$. Equation (7) can be shown to reduce to

$$P_{\text{nor}}(\theta) = \frac{2J_1(\beta R \sin\theta)}{\beta R \sin\theta} \tag{8}$$

in normalized form (see the Appendix). This is the well-known expression for the radiation pattern of a circular aperture with constant illumination (e.g., [16]).

III. Synthesis Method

It was shown above that a circular aperture can be represented by a collapsed equivalent line-source distribution. The question which now arises is whether the process can be reversed, or in other words, can one determine a rotationally symmetric aperture distribution $I(r)$ from a given line-source distribution? A method for doing so is to express $I(r)$ as a Fourier or Taylor series, or even a combination of both. The coefficients are then solved for by means of matrix inversion. Taking into account the typical distribution of $I(r)$, a suitable Fourier series expansion for $I(r)$ is given by

$$I(r) = a_0 + \sum_{n=1}^{N} a_n \cos\frac{n\pi r}{2R} + \sum_{n=1}^{N} b_n \sin\frac{n\pi r}{2R} \tag{9}$$

where R is the radius of the circular aperture and N is an arbitrary number. Substituting (9) into (5) and evaluating the result at $M = 2N+1$ points, one obtains M equations with M unknowns, namely

$$\begin{aligned} g(x_m) &= a_0\sqrt{R^2 - {x_m}^2} + \sum_{n=1}^{N} a_n \int_{0}^{\sqrt{R^2-{x_m}^2}} \cos\frac{n\pi r(x_m, y)}{2R}\, dy \\ &\quad + \sum_{n=1}^{N} b_n \int_{0}^{\sqrt{R^2-{x_m}^2}} \sin\frac{n\pi r(x_m, y)}{2R}\, dy \end{aligned} \tag{10}$$

where $0 \leq x_m \leq R, g(x)$ is the known line-source distribution and $m = 1 \rightarrow M$. The factor 2 in (5) was dropped, as it is a constant which will only influence the absolute values of the coefficients.

Equation (10) can be written in linear system matrix notation as

$$[A_{mn}][B_n] = [C_m] \tag{11}$$

where the excitation matrix is

$$[C_m] = [g(x_m)] \tag{12}$$

the matrix of unknown expansion coefficients is

$$[B_n] = \begin{bmatrix} a_{n-1}, & n = 1 \rightarrow N+1 \\ b_{n-N-1}, & n = N+2 \rightarrow 2N+1 \end{bmatrix} \tag{13}$$

and the coefficient matrix is given by (14), shown at the bottom of the page. In this paper, the matrix inversion to obtain B_n was done with the conjugate gradient method (CGM) [17], [18]. Once the coefficients in (13) have been determined, the rotationally symmetric radial distribution can be calculated from (9).

Instead of the Fourier series expansion, one can also use a Taylor series expansion to calculate $I(r)$, namely

$$I(r) = a_0 + \sum_{n=1}^{2N} a_n r^n. \tag{15}$$

In (15), the upper limit of the summation term was chosen as $2N$ for calculation purposes, to coincide with the Fourier expansion given by (9). The matrices for determining the coefficients are set up in similar fashion as the Fourier series approach. The coefficient matrix elements are given by

$$A_{mn} = \int_0^{\sqrt{R^2 - x_m{}^2}} \left[\frac{r(x_m, y)}{R}\right]^{n-1} dy, \quad n = 1 \rightarrow 2N+1. \tag{16}$$

Typical radial distributions will have a peak at the centre of the aperture and will then roll off toward the boundary, such as a cosine type of distribution. For such problems the cosine terms of (9) or the Taylor series of (15) will suffice. When the distribution magnitude near the boundary of the aperture rises again, as is typically the case for Taylor and Chebychev-type linear distributions, the sine terms of (9) become essential. One could combine all the terms of (9) and (15) to accommodate all possibilities, but the combination which was found to yield the most accurate results in general was

$$I(r) = a_0 + \sum_{n=1}^{K} a_n r^n + \sum_{n=1}^{N} b_n \sin\frac{n\pi r}{2R} \tag{17}$$

where K and N are not necessarily equal and the total number of unknowns is $M = K + N + 1$. The corresponding coefficient matrix is given by (18), shown at the bottom of the page.

IV. Examples

In order to check the accuracy of the synthesis method, a number of known rotationally symmetric radial distributions will be collapsed to equivalent line-source distributions which are a function of x only. These CELS distributions will then be used as the known $g(x)$ function and it will be tested whether the synthesis procedure will yield the same radial distribution as the original.

The simplest radial distribution to consider is a constant aperture illumination. The CELS distribution for a constant radial distribution is given by (6). This distribution was used as $g(x)$ and the number of unknowns M for all three expansions given by (9), (15), and (17) was selected to be 11. Table I presents the coefficients for all three expansions and Fig. 2 shows the normalized magnitude of the synthesized radial distributions on an expanded scale. All three expansions essentially yield the same result. It is worth noting that the coefficients of the Fourier series do not tend to zero very rapidly, and that the Fourier series by itself is the least accurate.

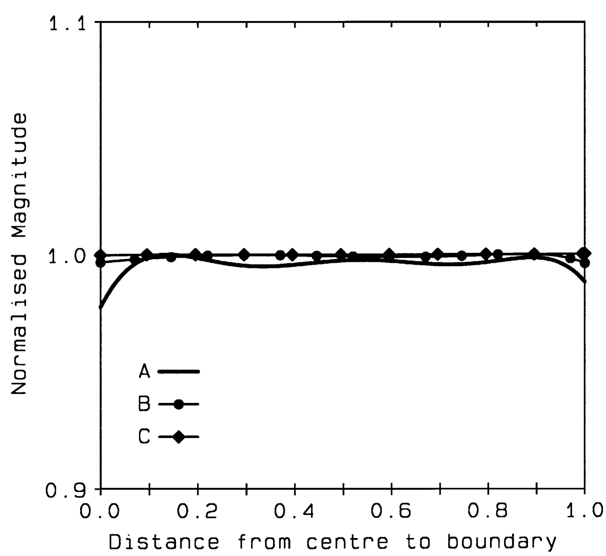
The next example tests the synthesis method for a linear radial distribution given by $I(r) = 1 - r/R$, also calculated with $M = 11$. The corresponding CELS distribution is shown in Fig. 3 (only the positive x-axis is shown) and the coefficients for the three expansions are given in Table I. Fig. 3 also shows

$$[A_{mn}] = \begin{bmatrix} \int_0^{\sqrt{R^2 - x_m{}^2}} \cos\frac{(n-1)\pi r(x_m,y)}{2R}\, dy, & n = 1 \rightarrow N+1 \\ \int_0^{\sqrt{R^2 - x_m{}^2}} \sin\frac{(n-N-1)\pi r(x_m,y)}{2R}\, dy, & n = N+2 \rightarrow 2N+1 \end{bmatrix}. \tag{14}$$

$$[A_{mn}] = \begin{bmatrix} \int_0^{\sqrt{R^2 - x_m{}^2}} \left[\frac{r(x_m,y)}{R}\right]^{n-1} dy, & n = 1 \rightarrow K+1 \\ \int_0^{\sqrt{R^2 - x_m{}^2}} \sin\frac{(n-K-1)\pi r(x_m,y)}{2R}\, dy, & n = K+2 \rightarrow K+N+1 \end{bmatrix}. \tag{18}$$

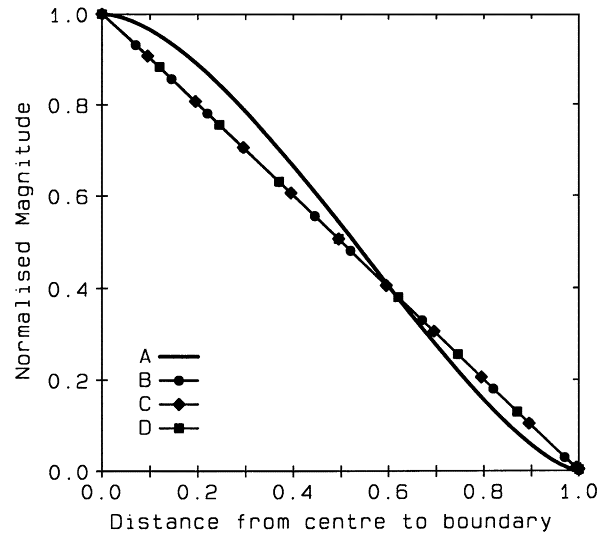
TABLE I
SERIES COEFFICIENTS FOR CONSTANT AND LINEAR RADIAL DISTRIBUTIONS

Constant illumination $I(r)=1$				Linear illumination $I(r)=1-r/R$			
n	Fourier	Taylor	Comb'd	n	Fourier	Taylor	Comb'd
1	+0.54	+1.00	+1.00	1	+0.67	+1.99	+2.00
2	+0.30	+0.02	+0.00	2	+0.70	-1.93	-0.43
3	-0.01	-0.03	-0.01	3	+0.61	-0.23	-0.34
4	+0.06	-0.01	+0.00	4	+0.19	+0.19	-0.24
5	+0.14	+0.01	+0.00	5	-0.10	+0.11	-0.16
6	-0.04	+0.02	+0.00	6	-0.07	-0.03	-0.09
7	+0.31	+0.01	+0.00	7	-0.09	-0.10	-0.02
8	+0.17	+0.01	+0.00	8	-0.02	-0.10	-0.54
9	-0.05	+0.00	+0.00	9	-0.04	-0.05	-0.40
10	+0.02	-0.01	+0.00	10	-0.35	+0.03	+0.17
11	-0.07	-0.02	+0.00	11	+0.01	+0.11	-0.04



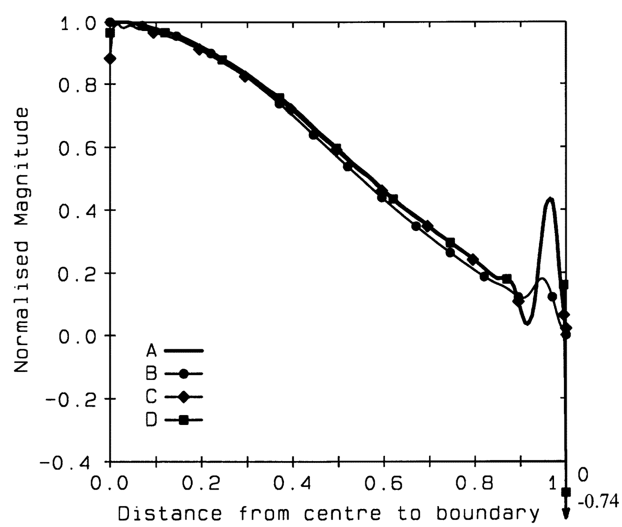
A Radial (Fourier) B Radial (Taylor) C Radial (Combined)

Fig. 2. Synthesized radial distributions for a uniformly illuminated circular aperture.



A CELS distribution B Radial (Fourier) C Radial (Taylor) D Radial (Combined)

Fig. 3. Synthesized radial solutions for a linear radial taper.



A Original/Radial (Combined) B CELS distribution C Radial (Fourier) D Radial (Taylor)

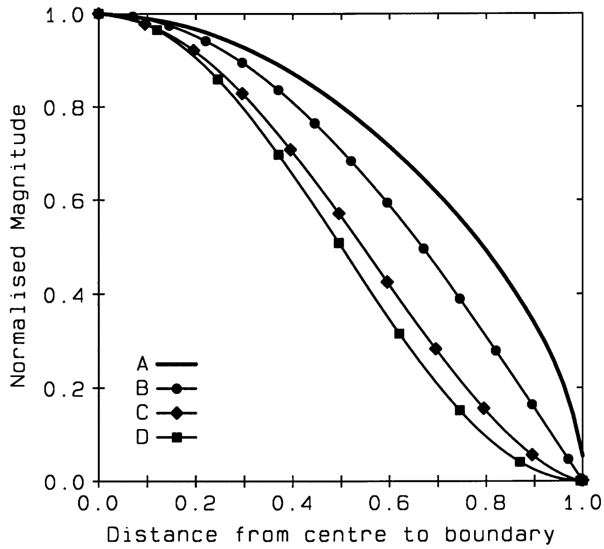
Fig. 4. Ludwig's CELS and synthesized radial distributions.

the synthesized radial distributions compared to the ideal linear radial distribution. The agreement between all three expansions is very good.

Fig. 4 shows the distribution function given by Ludwig [9] for a circular aperture with a sidelobe level of 40 dB, as well as the CELS distribution (Curve B). For this example there is a sharp rise in the illumination level toward the aperture boundary, so one might expect the series expansion with the sine terms to yield better accuracy than the Taylor expansion. This was indeed the case and the synthesized radial distributions can also be seen in Fig. 4 ($\mathrm{M} = 201, \mathrm{K} = 135$ for the combined approach). The Fourier expansion on its own is accurate over the entire range of x, except for the region where r is very small. The reason for this is that the arguments of the cosine and sine terms in (9) are very small, so that $I(r)$ is essentially given by the sum of the cosine coefficients. At $r = 0$ the cosine coefficients have to add up to zero, which does not happen since the cosine coefficients a_n do not rapidly tend to zero even for the larger values of n. The Taylor expansion is more accurate at $r = 0$, but has a large negative value at $r = R$ (value of -0.74). The distribution synthesized by the combined Fourier/Taylor expansion is virtually indistinguishable from the original distribution.

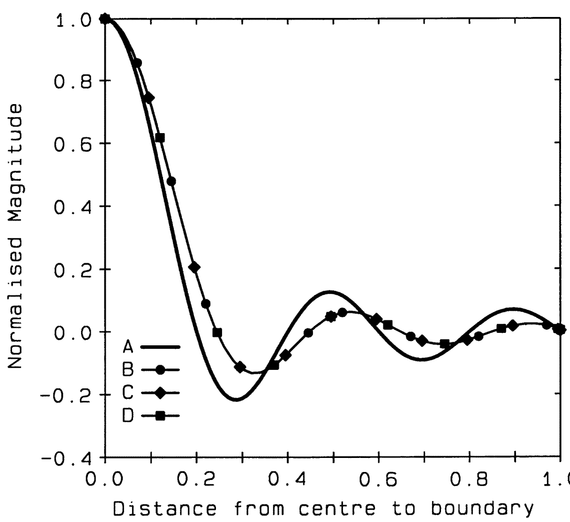
It should be noted that there seems to be no fixed rule for selecting the value of K (smaller or greater than $M/2$). It is best to experiment with the number of coefficients M as well as the break point K. When the error term in the CGM routine does not tend to zero, it means that more than one solution may be possible. This can usually be seen as a sharp drop or increase at $r = 0$ or $r = R$.

The preceding examples were presented to show that the method yields the correct radial distribution for collapsed known radial distributions. Proper synthesis from known line-source distributions will next be attempted. The first of these examples is the well-known cosine and cosine-squared



A COS linear B COS radial C COS² linear D COS² radial

Fig. 5. Cosine and cosine-squared distributions with associated radial distributions.

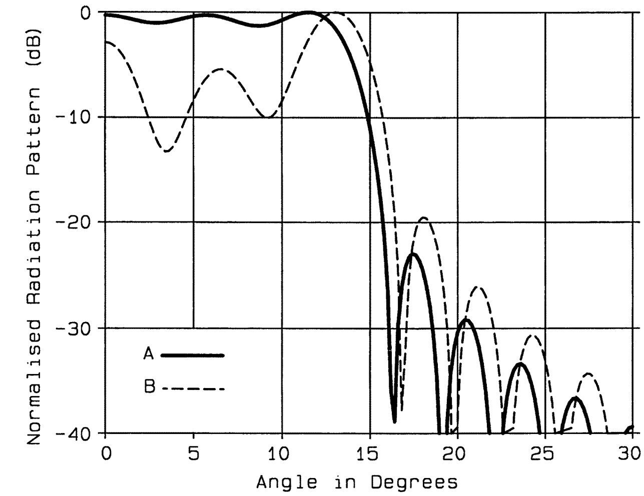


A Line-source distribution B Radial (Fourier) C Radial (Taylor) D Radial (Combined)

Fig. 6. $\mathrm{Sin}(x)/x$ line-source distribution with synthesized radial distributions.

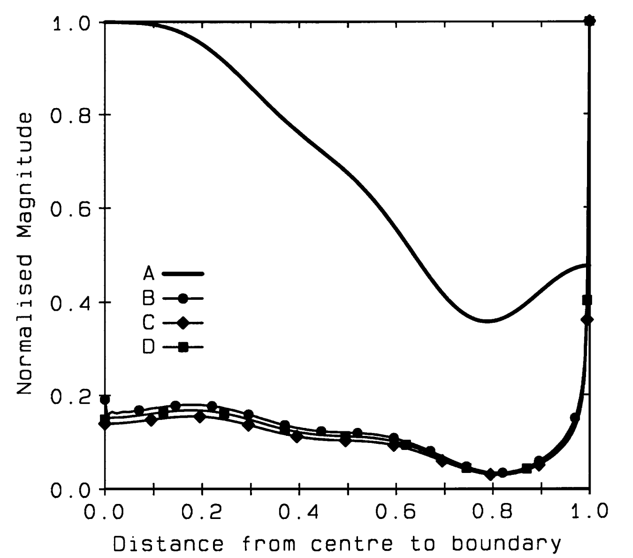
line-source distributions. Fig. 5 shows these distributions along with the corresponding radial distributions for a circular aperture (combined expansion, $M = 51, K = 35$). The accuracy of the synthesized radial distributions can easily be verified by calculation of the patterns of the circular apertures. The sidelobe levels of the synthesized radial distributions match those of the linear distributions summarized in ([16], Table 4-2). The sidelobe levels of the cosine and cosine-squared line-source distributions are -23.0 dB and -31.7 dB, respectively.

The second example is that of the $\sin(x)/x$ distribution shown in Fig. 6, which yields the pattern shown in Fig. 7. The associated radial distributions calculated by means of the three expansions types are also given in Fig. 6, with $M = 51$ and $K = 35$. Also shown in Fig. 7 is the radiation pattern that



A Linear and equivalent radial distributions B Linear distribution used as radial distribution

Fig. 7. Radiation patterns of $\sin(x)/x$ line-source distributions.



A Original linear B Radial (Fourier) C Radial (Taylor) D Radial (Combined)

Fig. 8. Elliott's line-source distribution example, with synthesized radial distributions.

would be obtained using the linear $\sin(x)/x$ distribution directly as the radial distribution. This is clearly a very inaccurate procedure to adopt.

The above example distributions are easily synthesized as the line-source illumination tends to zero at the edges. When the line-source distribution is not zero at the boundary, an exact solution cannot be obtained. The linear distribution in Fig. 8 (Curve A) was calculated by means of Elliott's perturbation technique [2] for a pattern with the height of the first sidelobe chosen to be -35 dB, the height of the next four -25 dB and the characteristic Taylor roll-off for the remainder of the sidelobes. This distribution cannot be regarded as a true CELS distribution of a circular aperture, as the edge illumination does not tend to zero. We will nevertheless attempt to synthesize an equivalent radial distribution which will yield a CELS distribution as shown in Fig. 8.

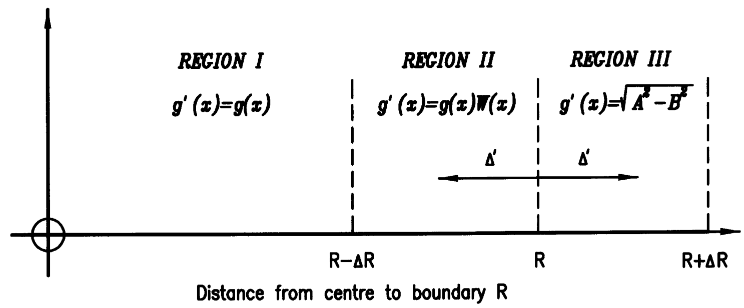


Fig. 9. Reduction of edge illumination by means of power spreading.

The three expansions, respectively, yield the radial distributions shown in Fig. 8 ($M = 201$ in all cases, $K = 135$ in combined formulation). The high value at $r = R$ is to be expected, as the CGM procedure will attempt to force a large nonzero illumination value where the integral forces it to be zero. When the radial distributions of Fig. 8 are collapsed to see how closely they represent the original line-source distribution, one obtains in all three cases CELS distributions which are a perfect match for the original. This implies that when the CGM routine does not convert to an exact solution (the error term does not tend to zero), the solution it converges to may not be unique.

In an attempt to obtain a more practical radial distribution for Elliott's distribution, the original distribution given in Fig. 8 is modified by a cosine weighting curve (a linear weighting function can also be used). The distribution is extended by ΔR, as shown in Fig. 9. The approach adopted here is based on the conservation of power, so that the total radiated power will in the end be the same as in the original case. The unmodified region (Region I) is defined as $0 \leq x \leq R - \Delta R$, and the new distribution $g'(x)$ is equal to the original line-source distribution, namely

$$g'(x) = g(x). \quad (19)$$

In the region $R - \Delta R < x \leq R$ (Region II) we define a weighting function

$$W(x) = \sqrt{(1 + \cos t)/2} \quad (20)$$

where

$$t = \frac{\pi}{2} \cdot \frac{x - (R - \Delta R)}{R - (R - \Delta R)}. \quad (21)$$

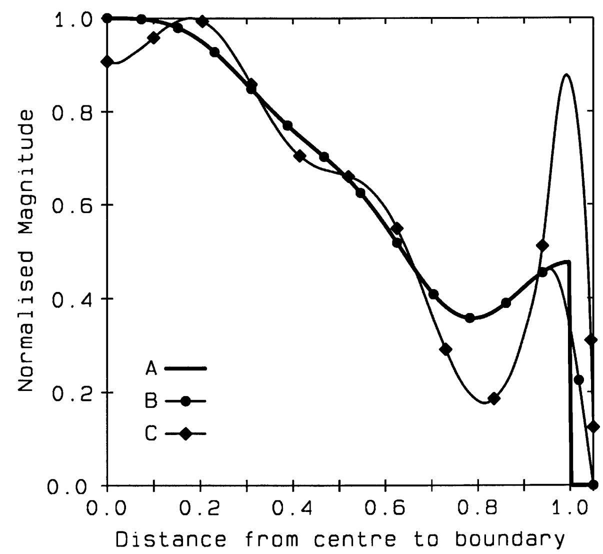
For $x = R - \Delta R, W = 1$. At $x = R$ we have $W = \sqrt{1/2}$, so that $W^2 = 0.5$. In the region $R - \Delta R < x \leq R$ we now define the new distribution $g'(x)$ as

$$g'(x) = g(x)W(x). \quad (22)$$

The power at the edge is thus reduced by a factor 2.

In the third region $R < x \leq R + \Delta R$ the original distribution is zero. We now fill the distribution in this region up with the power we have taken away from the region $R - \Delta R < x \leq R$, and we do so symmetrically around $x = R$. Defining $\Delta' = x - R$, we have in the third region

$$\begin{aligned} \mathrm{A} &= g(R - \Delta') \\ \mathrm{B} &= \mathrm{A}W(R - \Delta') \\ g'(x) &= \sqrt{\mathrm{A}^2 - \mathrm{B}^2}. \end{aligned} \quad (23)$$

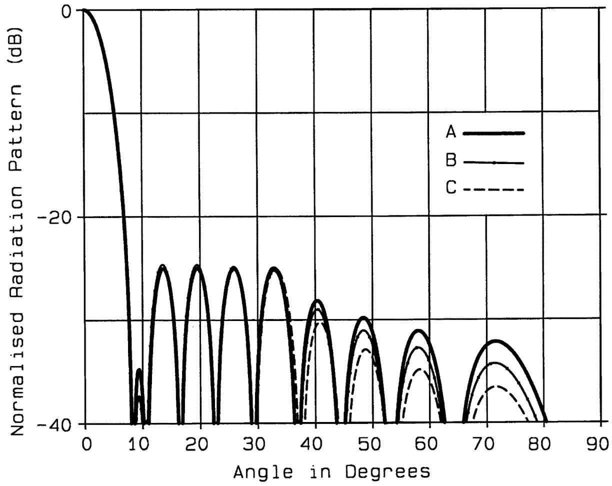


A Original linear B Modified linear C Equivalent radial (Combined)

Fig. 10. Elliott's linear distribution modified with $\Delta R = 0.05R$, and equivalent radial distribution.

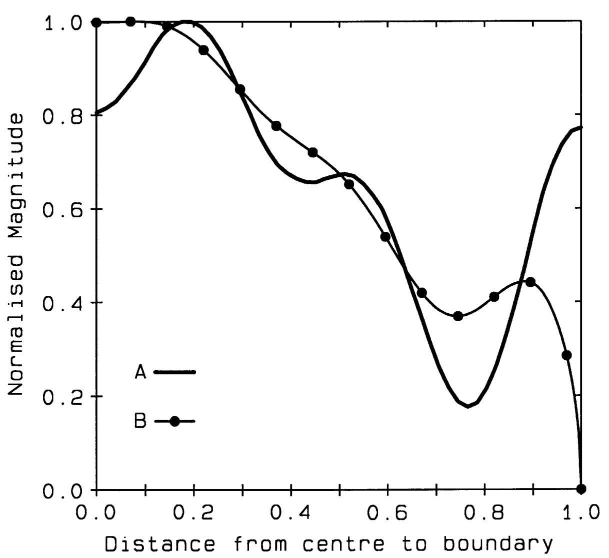
Fig. 10 shows the Elliott synthesis example's original linear distribution (Curve A) and the same distribution modified by a 5% power conservation window $\Delta R = 0.05R$ (Curve B). Fig. 11 shows the corresponding radiation patterns of the original distribution (Curve A, $R = 5\lambda$) and the modified distribution (Curve B, $R = 5.18\lambda$, adjusted to yield the same beamwidth as the original distribution). The modified distribution yields a pattern similar to the original pattern, with moderately distorted sidelobe levels. The first sidelobe is at -37.2 dB and the next three are slightly above -25 dB. By over-designing the original linear distribution, one can compensate for the distorted sidelobes to an extent. A "corrected" linear design for a first sidelobe at -33.2 dB and the following four at -25.4, -25.4, -25.2, and -25.0, respectively, converted by 5% power spreading, yields a pattern virtually indistinguishable from the original pattern (Fig. 11, Curve A), except for the far-out sidelobes which roll off at the same rate as Curve B.

With the modified distribution (Fig. 10, Curve B) next used as the CELS input for the new circular aperture synthesis method, one obtains the radial distribution calculated with the combined Taylor/sine series approach ($M = 201, K = 135$) as shown in Fig. 10, Curve C. This radial distribution is much more practical than those shown in Fig. 8.



A Original line-source distribution B Equivalent radial distribution for $\Delta R = 0.05R$ C [10, Case1]

Fig. 11. Radiation patterns for Elliott's line-source distributions.



A Radial distribution B CELS distribution

Fig. 12. Radial and CELS distributions for Elliott and Stern's circular aperture example.

As mentioned before, Elliott and Stern's method [10] is suitable for the design of circular aperture distributions with patterns of arbitrary shape and sidelobes of arbitrary individual height. The first example presented in ([10], Case 1) is for a circular aperture with a pattern shape identical to the pattern of the linear distribution discussed above (first sidelobe at −35 dB, next four at −25 dB, Taylor roll-off further on). The radial distribution for this example ([10], Case 1) is shown in Fig. 12 (Curve A). This distribution is significantly easier to implement than the radial distribution shown in Fig. 10 (Curve C) and must be regarded as the better of the two solutions. When the radial distribution of Fig. 12 (Curve A) is collapsed, the CELS distribution given by Curve B in Fig. 12 is obtained. It can easily be verified that for this CELS distribution, the new method (combined expansion) yields exactly the same radial distribution as Fig. 12, Curve A. An instructive observation is that the CELS distribution has zero edge illumination. Classic linear distribution synthesis, even by Elliott's original method (Fig. 8, Curve A), does not allow the restriction of zero edge illumination to be imposed. Although the *radial* distribution for a given pattern shape and sidelobe envelope does not necessarily have zero edge illumination, the collapsed (linear) distribution always will. One can therefore use the circular aperture synthesis method of Elliott and Stern to synthesize linear distributions for arbitrarily shaped symmetric patterns with the additional constraint of zero edge illumination.

The radiation pattern of a circular aperture with the radial distribution of Curve A in Fig. 12 is shown overlaid (Curve C) with the linear distribution patterns in Fig. 11 ($R = 5.13\lambda$ for achieving the same beamwidth as the other patterns). The specified sidelobes of the circular aperture pattern are almost indistinguishable from the specified sidelobes of the linear distribution, but the far-out sidelobes roll off at a faster rate than for the linear distribution.

V. Conclusion

A technique was presented which allows the distribution of a circular aperture antenna to be synthesized to have the same rotationally symmetric radiation pattern as the principal plane pattern of a known line-source distribution. A new degree of freedom is thus introduced in the synthesis of circular aperture antenna patterns. The line-source distribution must be symmetrical with respect to the centre and should ideally have zero edge illumination. Although no proof of the uniqueness of the synthesized radial distributions was given, various examples showed that the synthesis procedure produces the correct rotationally symmetric radial distribution when the CELS distribution has zero edge illumination. If the CELS edge illumination is not zero, the solution may no longer be unique and the three expansion functions used in the paper may yield different radial distributions as a solution. These radial distributions nevertheless yield the same CELS distribution when the synthesized radial distribution is collapsed again to see how well it approximates the target line-source distribution. In practice one should experiment with the number of unknowns M, the approach (Fourier, Taylor or the combination), the relative value of K and if necessary, the amount of power spreading, to find the optimal solution.

Although power spreading is an option for line-source distributions with nonzero edge illumination, it is in this case probably better to use the generalized synthesis method for circular apertures as proposed by [10]. This generalized synthesis method can be used to derive symmetric line-source distributions with the additional constraint of zero edge illumination, for patterns of arbitrary shape and sidelobe envelopes.

Appendix

The radiation pattern of an isotropic line-source distribution $I(x)$ of length $2R$ is given by (e.g., [16])

$$P(\theta) = \int_{-R}^{R} I(x) e^{j\beta x \sin\theta}\, dx \tag{24}$$

where $\beta = 2\pi/\lambda$ and $\theta = 0°$ is the broadside direction of the line-source. For a circular aperture with constant illumination the CELS distribution is given by (6) and (24) becomes

$$P(\theta) = \int_{-R}^{R} \sqrt{R^2 - x^2} e^{j\beta x \sin\theta}\, dx. \qquad (25)$$

Expressing the exponential term in (25) in $\cos + j \sin$ notation and recognising that the integral of the sine term will be zero, (25) can be written as

$$\begin{aligned} P(\theta) &= \int_{-R}^{R} \sqrt{R^2 - x^2} \cos(\beta x \sin\theta)\, dx \\ &= 2\int_{0}^{R} \sqrt{R^2 - x^2} \cos(\beta x \sin\theta)\, dx. \end{aligned} \qquad (26)$$

Performing the substitution $x = tR$ in (26) results in

$$\begin{aligned} P(\theta) &= 2R^2 \int_{0}^{1} \sqrt{1 - t^2} \cos(\beta R t \sin\theta)\, dt \\ &= 2R^2 \int_{0}^{1} \sqrt{1 - t^2} \cos(zt)\, dt \end{aligned} \qquad (27)$$

where $z = \beta R \sin\theta$. From [19] we have the relation

$$J_\nu(z) = \frac{2\left(\frac{1}{2}z\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_{0}^{1} (1 - t^2)^{\nu - \frac{1}{2}} \cos(zt)\, dt \qquad (28)$$

where $J_\nu(z)$ is a Bessel function of order ν. For $\nu = 1$ we have $\Gamma(3/2) = (1/2)\sqrt{\pi}$ and (27) reduces to

$$P(\theta) = 2\pi R \frac{J_1(\beta R \sin\theta)}{\beta \sin\theta}. \qquad (29)$$

This expression is identical in form to [16, (8–135)], where the radiation pattern for a uniform circular aperture was derived from first principles. Equation (29) can be normalized for a maximum of unity at $\theta = 0°$, which then becomes

$$P_{\text{nor}}(\theta) = \frac{2J_1(\beta R \sin\theta)}{\beta R \sin\theta} \qquad (30)$$

as was done in [16].

Acknowledgment

The author is indebted to Dr. E. Botha for helpful discussions along the way and to the reviewers for suggesting several improvements to the paper.

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